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PRECESSIONAL-ISOCONIC MOTION OF A RIGID BODY WITH A FIXED POINT†

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A class of precessional-isoconic motions of a gyrostat with a fixed point is considered in the generalized dynamical problem. New classes of such motions are found, where the precession of the body can either be a semi-regular precession of the second type, or a precessional motion of general form.

THE INVESTIGATION of precessional-isoconic motion is an important stage in the study of precession. In this case, the motion possesses the property of isoconicity (the moving hodograph of the angular velocity vector being congruent with the hodograph fixed relative to the tangent plane) as well as that of precession. Isoconic motions in dynamics were apparently first considered by Fabbri [1] who established their existence in the well-known Steklov solution. Using the hodograph method [2] this property was found in [3]. Apart from this case, isoconic motions have been observed in the solutions of Lagrange, Zhukovskii [4], Hess–Sretenskii [4] and Grioli [5]. All these investigations concern the classical problem of the motion of a gyrostat in a gravitational field. In the generalized dynamical problem a result is known [6] concerning the conditions for the existence of isoconic gyrostat motion with the first level of the appropriate invariant relation.

1. STATEMENT OF THE PROBLEM

Consider the generalized problem of the motion of a gyrostat with a fixed point. We will write the equations of motion in the form [7, 8]

$$A\,\dot{\boldsymbol{\omega}} = (A\,\boldsymbol{\omega} + \boldsymbol{\lambda}) \times \boldsymbol{\omega} + \boldsymbol{\omega} \times B\boldsymbol{\nu} + \mathbf{s} \times \boldsymbol{\nu} + \boldsymbol{\nu} \times C\boldsymbol{\nu} \tag{1.1}$$

$$\dot{\boldsymbol{\nu}} = \boldsymbol{\nu} \times \boldsymbol{\omega} \tag{1.2}$$

They admit of the first integrals

$$\begin{aligned} &A\,\boldsymbol{\omega}\cdot\boldsymbol{\omega} - 2\,(\mathbf{s}\cdot\boldsymbol{\nu}) + C\boldsymbol{\nu}\cdot\boldsymbol{\nu} = 2\,E, \quad \boldsymbol{\nu}\cdot\boldsymbol{\nu} = 1 \\ &(A\,\boldsymbol{\omega} + \boldsymbol{\lambda})\cdot\boldsymbol{\nu} - \tfrac{1}{2}\,(B\boldsymbol{\nu}\cdot\boldsymbol{\nu}) = k \end{aligned} \tag{1.3}$$

In (1.1)–(1.3) $\boldsymbol{\omega}$ is the angular velocity of the gyrostat, $\boldsymbol{\nu}$ is the unit vector describing the direction of the axis of symmetry of the force field, $\boldsymbol{\lambda}$ is the gyrostatic moment, $\mathbf{s}$ is the generalized centre of mass vector, A is the inertia tensor of the gyrostat constructed at the fixed point, and B and C are symmetric 3×3 matrices [7, 8]. A dot above the variables denotes the

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derivative with respect to time.

Suppose the gyrostat motion is precessional about the vertical (the angle between the unit vector $\mathbf{a}$ fixed to the body and the vector $\boldsymbol{\nu}$ is constant). We then have the invariant relation [9]

$$\boldsymbol{\nu}\cdot\mathbf{a}=a_0,\quad a_0=\cos\theta_0 \tag{1.4}$$

where θ_0 is the angle between $\mathbf{a}$ and $\boldsymbol{\nu}$. From (1.2) the derivative of (1.4) gives $\boldsymbol{\omega}\cdot(\mathbf{a}\times\boldsymbol{\nu})=0$, i.e.

$$\boldsymbol{\omega}=f_1(t)\,\mathbf{a}+f_2(t)\,\boldsymbol{\nu} \tag{1.5}$$

The case $\mathbf{a}\times\boldsymbol{\nu}=0$ is excluded because it implies that the gyrostat rotates uniformly. Substituting (1.5) into Eq. (1.2) we obtain

$$\boldsymbol{\nu}^{\boldsymbol{\cdot}}=f_1(t)(\boldsymbol{\nu}\times\mathbf{a}) \tag{1.6}$$

We attach to the body a moving system of coordinates such that the vector $\mathbf{a}$ has the form $\mathbf{a}=(0,\ 0,\ 1)$. We then satisfy Eqs (1.4), $\boldsymbol{\nu}\cdot\boldsymbol{\nu}=1$ and (1.6) by introducing a new variable φ

$$\boldsymbol{\nu}=(a_0'\sin\varphi,\ a_0'\cos\varphi,\ a_0),\quad a_0'=\sin\theta_0 \tag{1.7}$$

and putting $f_1(t)=\varphi^{\boldsymbol{\cdot}}$. The variable φ plays the role of the angle of proper rotation of the gyrostat. If ψ denotes its angle of precession, then in (1.5) $f_2(t)=\psi^{\boldsymbol{\cdot}}$ and so

$$\boldsymbol{\omega}=\varphi^{\boldsymbol{\cdot}}\mathbf{a}+\psi^{\boldsymbol{\cdot}}\boldsymbol{\nu} \tag{1.8}$$

Substituting (1.8) into (1.1) and (1.3) we obtain

$$\begin{aligned}&\varphi^{\boldsymbol{\cdot\cdot}}A\,\mathbf{a}+\psi^{\boldsymbol{\cdot\cdot}}A\,\boldsymbol{\nu}+\varphi^{\boldsymbol{\cdot}}\psi^{\boldsymbol{\cdot}}\,[\mathrm{Tr}\,(A)(\boldsymbol{\nu}\times\mathbf{a})-2\,(A\,\boldsymbol{\nu}\times\mathbf{a})]-\\&-\varphi^{\boldsymbol{\cdot}2}(A\,\mathbf{a}\times\mathbf{a})-\psi^{\boldsymbol{\cdot}2}(A\,\boldsymbol{\nu}\times\boldsymbol{\nu})+\varphi^{\boldsymbol{\cdot}}\mathbf{a}\times(\boldsymbol{\lambda}-B\,\boldsymbol{\nu})+\\&+\psi^{\boldsymbol{\cdot}}\boldsymbol{\nu}\times(\boldsymbol{\lambda}-B\,\boldsymbol{\nu})-\mathbf{s}\times\boldsymbol{\nu}-\boldsymbol{\nu}\times C\boldsymbol{\nu}=0\end{aligned} \tag{1.9}$$

$$\varphi^{\boldsymbol{\cdot}}(A\,\mathbf{a}\cdot\boldsymbol{\nu})+\psi^{\boldsymbol{\cdot}}(A\,\boldsymbol{\nu}\cdot\boldsymbol{\nu})=k-\boldsymbol{\lambda}\cdot\boldsymbol{\nu}+{}^1\!/_2\,(B\,\boldsymbol{\nu}\cdot\boldsymbol{\nu}) \tag{1.10}$$

$$\begin{aligned}&\varphi^{\boldsymbol{\cdot}2}(A\,\mathbf{a}\cdot\mathbf{a})+2\,\varphi^{\boldsymbol{\cdot}}\psi^{\boldsymbol{\cdot}}(A\,\mathbf{a}\cdot\boldsymbol{\nu})+\psi^{\boldsymbol{\cdot}2}(A\,\boldsymbol{\nu}\cdot\boldsymbol{\nu})=\\&=2\,(E+\mathbf{s}\cdot\boldsymbol{\nu})-C\boldsymbol{\nu}\cdot\boldsymbol{\nu}.\end{aligned}$$

Since the vectors $\mathbf{a},\boldsymbol{\nu}$ and $\mathbf{a}\times\boldsymbol{\nu}$ are independent, we consider the projections of the left-hand side of (1.9) along these vectors. It can be shown that the projections along the vectors $\mathbf{a}$ and $\boldsymbol{\nu}$ reduce to Eq. (1.10), so we only write down the projection along $\mathbf{a}\times\boldsymbol{\nu}$

$$\begin{aligned}&\varphi^{\boldsymbol{\cdot\cdot}}A\,\mathbf{a}\cdot(\boldsymbol{\nu}\times\mathbf{a})+\psi^{\boldsymbol{\cdot\cdot}}A\,\boldsymbol{\nu}\cdot(\boldsymbol{\nu}\times\mathbf{a})+\varphi^{\boldsymbol{\cdot}}\psi^{\boldsymbol{\cdot}}\,[\mathrm{Tr}\,(A)\,a_0'^2-\\&-2\,(A\,\boldsymbol{\nu}\cdot\boldsymbol{\nu})+2\,a_0\,(A\,\mathbf{a}\cdot\boldsymbol{\nu})]+\varphi^{\boldsymbol{\cdot}2}[a_0\,(A\,\mathbf{a}\cdot\mathbf{a})-\\&-A\,\mathbf{a}\cdot\boldsymbol{\nu}]-\psi^{\boldsymbol{\cdot}2}[a_0\,(A\,\boldsymbol{\nu}\cdot\boldsymbol{\nu})-A\,\mathbf{a}\cdot\boldsymbol{\nu}]+\varphi^{\boldsymbol{\cdot}}[a_0\,(\boldsymbol{\lambda}\cdot\mathbf{a})-\\&-\boldsymbol{\lambda}\cdot\boldsymbol{\nu}-a_0\,(B\,\mathbf{a}\cdot\boldsymbol{\nu})+B\,\boldsymbol{\nu}\cdot\boldsymbol{\nu}]+\psi^{\boldsymbol{\cdot}}[\boldsymbol{\lambda}\cdot\mathbf{a}-a_0\,(\boldsymbol{\lambda}\cdot\boldsymbol{\nu})-B\,\mathbf{a}\cdot\boldsymbol{\nu}+\\&+a_0\,(B\,\boldsymbol{\nu}\cdot\boldsymbol{\nu})]+\mathbf{a}\cdot\mathbf{s}-a_0\,(\mathbf{s}\cdot\boldsymbol{\nu})+a_0\,(C\boldsymbol{\nu}\cdot\boldsymbol{\nu})-C\,\mathbf{a}\cdot\boldsymbol{\nu}=0\end{aligned} \tag{1.11}$$

The method for investigating precession [9] about the vertical is as follows. From (1.20) we find the dependence of $\varphi^{\boldsymbol{\cdot}}$ and $\psi^{\boldsymbol{\cdot}}$ on ψ and the problem parameters. Substituting these expressions into Eq. (1.11) we obtain an equation of the form $F(\varphi,\ \boldsymbol{\lambda},\ A,\ B,\ C,\ k,\ E)=0$. The requirement that it be an identity in φ imposes conditions on the parameters whose satisfaction leads to precession of the motion of the body about the vertical.

Suppose that the motion of the gyrostat has the isoconicity property [9] as well as precession.

Then we have the additional invariant relation

$$\boldsymbol{\omega}\cdot(\boldsymbol{\nu}-\mathbf{c})=0 \tag{1.12}$$

where **c** is a unit vector fixed to the body (and in general different from **a**). It can be shown that when (1.12) is satisfied the moving hodograph of the angular velocity vector is congruent with the fixed one [9].

We substitute (1.8) into (1.12)

$$\varphi^{\cdot}(a_0-\mathbf{a}\cdot\mathbf{c})+\psi^{\cdot}(1-\boldsymbol{\nu}\cdot\mathbf{c})=0 \tag{1.13}$$

Thus, in addition to (1.10) and (1.11) we obtain yet another condition on $\varphi^{\cdot}$ and $\psi^{\cdot}$. Without loss of generality we shall specify the vector **c** in the form $c=(c_1,\ 0,\ c_3)$, where $c_1^2+c_3^2=1$.

2. REGULAR PRECESSIONAL-ISOCONIC MOTIONS

Suppose the gyrostat precession is regular $\varphi^{\cdot}=n,\ \psi^{\cdot}=m$. Here relation (1.13) should be an identity in φ and so $c_1=0\ \ (c=\mathbf{a})$ and $n=m$. Consequently, the condition for the existence of precessional-isoconic motion for the case when the precession is regular is found from the conditions for regular precessions to exist [10] having put $m=n$

$$\begin{aligned}
&B_{12}=0,\quad C_{12}=0,\quad 2n(A_{22}-A_{11})-B_{22}+B_{11}=0\\
&n^2(A_{22}-A_{11})+C_{22}-C_{11}=0,\quad n^2A_{13}-nB_{13}-C_{13}=0\\
&n^2A_{13}-nB_{23}-C_{23}=0,\quad s_1=a_0C_{13}+n^2A_{13}(a_0+1)\\
&s_2=a_0C_{23}+n^2A_{23}(a_0+1),\quad \lambda_1=B_{13}a_0-A_{13}n(2a_0+1)\\
&\lambda_2=B_{23}a_0-A_{23}n(2a_0+1),\quad n^2(A_{22}+A_{33}-A_{11})+\\
&+n\lambda_3+B_{11}n(a_0+1)-B_{33}na_0-a_0n^2(A_{11}-A_{33})+\\
&+a_0(C_{11}-C_{33})+s_3=0.
\end{aligned} \tag{2.1}$$

The moving hodograph of the angular velocity vector is given by the relations

$$\omega_1=na_0'\sin\varphi,\quad \omega_2=na_0'\cos\varphi,\quad \omega_3=n(1+a_0) \tag{2.2}$$

where $\varphi=nt+\varphi_0$, φ_0 is an arbitrary constant. On the basis of the Kharlamov equations [2] we also find the equation of the stationary hodograph in cylindrical coordinates

$$\omega_\xi=n(1+a_0),\quad \omega_\rho=|a_0'n|,\quad \alpha=nt+\alpha_0 \tag{2.3}$$

It follows from (2.2) and (2.3) that the motion of the body is periodic with period $T=2\pi/|n|$.

3. SEMIREGULAR PRECESSIONAL-ISOCONIC MOTION OF THE FIRST TYPE

We will consider the case when the motion of the body is a semiregular precession of the first type: $\psi^{\cdot}=m,\ \varphi^{\cdot}\neq\text{const}$. Obviously, in (1.13) $\mathbf{c}\neq\mathbf{a}$ because otherwise the precession is regular. Introducing the new parameters

$$b_0 = \frac{a_0 c_3 - 1}{a_0 - c_3}, \quad c_0 = \frac{a_0' c_1}{a_0 - c_3} \tag{3.1}$$

which obviously satisfy the equation $b_0^2 = 1 + c_0^2$, from (1.13) we obtain

$$\varphi^{\bullet} = m\,(b_0 + c_0 \sin\varphi) \tag{3.2}$$

Semiregular precession of the firsts type with proper rotational velocity of the form (3.2) for the generalized dynamical problem was studied in [11], where conditions for this type of precession to exist were written as equalities which must be satisfied by the parameters in the system of equations (1.1), (1.2). If one additionally requires that the condition $b_0^2 = 1 + c_0^2$, is satisfied, the gyrostat motion has the properties of isoconicity and semiregular precession of the first type. When b_0 and c_0 are known, the quantities c_1 and c_3 are given from (3.1) by the formulae

$$c_1 = -\frac{c_0 a_0'}{a_0 + b_0}, \quad c_3 = \frac{a_0 b_0 + 1}{a_0 + b_0}$$

We conclude from the equality $b_0^2 = 1 + c_0^2$ that φ is a monotonic function of time. To fix our ideas we will put $m > 0$, $b_0 > 0$, $c_0 > 0$ and we find from (3.2) that

$$\frac{\varphi}{2} = \operatorname{arctg}\left[b_0 \operatorname{tg}\frac{mt}{2}\left(1 - c_0 \operatorname{tg}\frac{mt}{2}\right)^{-1}\right] \tag{3.3}$$

The equations of the moving hodograph are

$$\omega_1 = m a_0' \sin\varphi, \quad \omega_2 = m a_0' \cos\varphi, \quad \omega_3 = m\,[a + b_0 + c_0 \sin\varphi] \tag{3.4}$$

It follows from (3.3) and (3.4) that the moving hodograph is the curve of the intersection of the cylinder $\omega_1^2 + \omega_2^2 = a_0'^2 m^2$ with the plane $\omega_3 = (c_0 / a_0')\omega_1 + m(a_0 + b_0)$, and the tip of the angular velocity vector moves along it periodically with period $T = 2\pi / m$.

We write the equation of the stationary hodograph in the Cartesian system of coordinates

$$\omega_\xi = \omega_\rho \cos\alpha, \quad \omega_\eta = \omega_\rho \sin\alpha, \quad \omega_\zeta :$$

$$\omega_\xi = a_0' m b_0^{-1}\,(c_0^2 + \cos\varphi + b_0 c_0 \sin\varphi)$$

$$\omega_\eta = a_0' m b_0^{-1}\,(c_0 - c_0 \cos\varphi + b_0 \sin\varphi)$$

$$\omega_\zeta = m\,(1 + a_0 b_0 + a_0 c_0 \sin\varphi)$$

Because of the isoconicity the motion of the body is periodic with period $2\pi / m$.

4. SEMIREGULAR PRECESSIONAL-ISOCONIC MOTIONS OF THE SECOND TYPE

Suppose that in relations (1.10), (1.11) and (1.13) $\varphi^{\bullet} = n$. Then (1.13) gives

$$\psi^{\bullet} = n / \Delta(\varphi), \quad \Delta(\varphi) = b_0 + c_0 \sin\varphi \tag{4.1}$$

and it follows from (1.10) and (1.11) that

$$n\,(d_1 \cos\varphi + d_1' \sin\varphi + d_0) + \psi^{\bullet}(a_2 \cos 2\varphi + a_2' \sin 2\varphi +$$
$$+ a_1 \cos\varphi + a_1' \sin\varphi + a_0^*) - (b_1 \cos 2\varphi + b_2' \sin 2\varphi +$$
$$+ b_1 \cos\varphi + b_1' \sin\varphi + b_0^*) = 0 \tag{4.2}$$

$$n^2A_{33}+2n\dot\psi(d_1\cos\varphi+d_1'\sin\varphi+d_0)+\dot\psi^2(a_2\cos 2\varphi+ a_2'\sin 2\varphi+a_1\cos\varphi+a_1'\sin\varphi+a_0^*)-(c_2\cos 2\varphi+ c_2'\sin 2\varphi+c_1\cos\varphi+c_1'\sin\varphi+c_0^*)=0 \tag{4.3}$$

$$\ddot\psi(-a_2\sin 2\varphi+a_2'\cos 2\varphi+{}^1\!/_2\, a_1'\cos\varphi-{}^1\!/_2\, a_1\sin\varphi)+ n\dot\psi(-2a_2\cos 2\varphi-2a_2'\sin 2\varphi-a_1\cos\varphi-a_1'\sin\varphi+ d_0')-n^2(d_1\cos\varphi+d_1'\sin\varphi)-\dot\psi^2(a_2a_0\cos 2\varphi+ a_2'a_0\sin 2\varphi+p_1\cos\varphi+p_1'\sin\varphi+p_0)+n(2b_0\cos 2\varphi+ 2b_2'\sin 2\varphi+b_1\cos\varphi+b_1'\sin\varphi+b_0^{**})+\dot\psi(2b_2a_0\cos 2\varphi+ 2b_2'a_0\sin 2\varphi+q_1\cos\varphi+q_1'\sin\varphi+q_0)-a_0c_2\cos 2\varphi- -a_0c_2'\sin 2\varphi+r_1\cos\varphi+r_1'\sin\varphi+r_0=0 \tag{4.4}$$

where the following notation has been introduced

$$a_2={}^1\!/_2\, a_0'^2(A_{22}-A_{11}),\quad a_2'=a_0'^2A_{12},\quad a_1=2a_0a_0'A_{23}$$
$$a_1'=2a_0a_0'A_{13},\quad a_0^*={}^1\!/_2\, a_0'^2(A_{22}+A_{11})+a_0^2A_{33}$$
$$d_0=a_0A_{33},\quad d_1=a_0'A_{23},\quad d_1'=a_0'A_{13}$$
$$b_2={}^1\!/_2\, a_0'^2(B_{22}-B_{11}),\quad b_2'={}^1\!/_2\, a_0'^2B_{12}$$
$$b_1=a_0'(B_{23}a_0-\lambda_2),\quad b_1'=a_0'(B_{13}a_0-\lambda_1)$$
$$b_0^*=k-\lambda_3a_0+{}^1\!/_4\, a_0^2(B_{11}+B_{22})+{}^1\!/_2\, a_0^2B_{33}$$
$$b_0^{**}={}^1\!/_2\, a_0'^2(B_{11}+B_{22}),\quad c_2={}^1\!/_2\, a_0'^2(C_{11}-C_{22})$$
$$c_2'=-a_0'^2C_{12},\quad c_1=2a_0'(s_2-C_{23}a_0)$$
$$c_1'=2a_0'(s_1-C_{13}a_0),\quad c_0^*=2E+2s_3a_0-{}^1\!/_2(C_{11}+C_{22})a_0'^2-C_{33}a_0^2$$
$$p_1=a_0'A_{23}(a_0^2-a_0'^2),\quad p_1'=a_0'A_{13}(a_0^2-a_0'^2)$$
$$d_0'=a_0'^2A_{33},\quad p_0={}^1\!/_2\, a_0a_0'^2(A_{11}+A_{22}-2A_{33})$$
$$q_1=a_0'[B_{23}(a_0^2-a_0'^2)-a_0\lambda_2],\quad q_1'=a_0'[B_{13}(a_0^2-a_0'^2)-a_0\lambda_1]$$
$$q_0=a_0'^2[{}^1\!/_2\, a_0(B_{11}+B_{22}-2B_{33})+\lambda_3],\quad r_1=a_0'[C_{23}(a_0^2-a_0'^2)-a_0s_2]$$
$$r_1'=a_0'[C_{13}(a_0^2-a_0'^2)-a_0s_1]$$
$$r_0=a_0'^2[{}^1\!/_2\, a_0(C_{11}+C_{22}-2C_{33})+s_3]$$

We substitute (4.1) into (4.2)–(4.4) and require the resulting equations to be identities in φ. We find as a result a non-linear system of algebraic equations connecting the system parameters.

After calculations one can show that it has the following solution

$$A_{12}=A_{23}=0,\quad B_{12}=B_{23}=0,\quad B_{11}=B_{22},\quad C_{12}=C_{13}=C_{23}=0$$
$$C_{11}=C_{22},\quad s_1=s_2=0,\quad \lambda_2=0,\quad \lambda_1={}^1\!/c_0\,[c_0(B_{13}a_0- -nA_{13})+na_0'(A_{22}-A_{11})],\quad s_3=a_0(C_{33}-C_{11})+$$
$$+\frac{n}{c_0}(B_{13}a_0'-B_{11}c_0),\quad \lambda_3a_0'^2=a_0a_0'^2(B_{33}-B_{11})+$$
$$+\frac{1}{nc_0b_0}[na_0c_0^2A_{13}-nb_0c_0a_0'(A_{11}-A_{22}+A_{33})+ +na_0a_0'c_0(A_{22}-A_{33})-b_0^2a_0'^2B_{13}],\quad b_0^2=\frac{1}{1-\lambda^2} \tag{4.5}$$

$$c_0^2 = \frac{\lambda^2}{1-\lambda^2}, \quad \lambda^2 (a_0'^2 A_{22} + a_0^2 A_{33}) - 2\lambda a_0 a_0' A_{13} -$$
$$-a_0'^2 (A_{22} - A_{11}) = 0, \quad (A_{22} + \sigma A_{33})(Q_1\sigma + Q_0)^2 -$$
$$-4\sigma A_{13}^2 (R_1\sigma + R_0)(Q_1\sigma + Q_0) - 4\sigma A_{13}^2 (A_{22} - A_{11})(R_1\sigma + R_0)^2 = 0$$

Here

$$\sigma = a_0^2/a_0'^2, \quad Q_1 = A_{13}^2 (A_{22} - A_{11} + A_{33}) + A_{33}(A_{22} - A_{11})(A_{33} - A_{11} - A_{22})$$
$$Q_0 = A_{22}[A_{13}^2 - A_{11}(A_{22} - A_{11})]$$
$$R_1 = A_{11}A_{33} - A_{13}^2, \quad R_0 = A_{22}(A_{11} - A_{22} + A_{33})$$

The last equation has a solution for σ, for example, for the following values: $A_{11} = 2a$, $A_{22} = 3a$, $A_{33} = 4a$, $A_{13} = a$, where a is an arbitrary parameter, because $f(0) > 0$, $f(\infty) < 0$ (here $f(\sigma)$ is the left-hand side of the equation under consideration). Obviously, here the penultimate equation of system (4.4) has a solution with respect to λ. Thus, the solvability of system (4.5) is proved, and we have therefore established conditions for precessional-isoconic motion of the second type to exist.

The moving hodograph of the angular velocity vector is given by the equations

$$\omega_1 = a_0' n \sin nt/\Delta(nt), \quad \omega_2 = a_0' n \cos nt/\Delta(nt)$$
$$\omega_3 = n(1 + a_0/\Delta(nt)) \tag{4.6}$$

and is therefore the line of intersection of an elliptical cylinder with the cone

$$\frac{(\omega_1 + a_0' c_0 n)^2}{a_0'^2 n^2 b_0^2} + \frac{\omega_2^2}{a_0'^2 n^2} = 1, \quad \omega_1^2 + \omega_2^2 - \frac{a_0'^2 n^2}{a_0^2}(\omega_3 - n)^2 = 0$$

The stationary hodograph is given by the relations

$$\omega_\xi = n(a_0 + 1/\Delta(nt)), \quad \omega_\rho = |a_0' n|$$
$$\alpha = 2 \operatorname{arctg}\left[\operatorname{tg}\frac{nt}{2}\left(b_0 + c_0 \operatorname{tg}\frac{nt}{2}\right)^{-1}\right] \tag{4.7}$$

It follows from (4.6), (4.7) that the motion of the body is periodic with period $2\pi/n$. The congruency of the moving and stationary hodograph is obvious.

5. PRECESSIONAL-ISOCONIC MOTION OF THE GENERAL TYPE

Suppose that $\varphi^\cdot$ and $\psi^\cdot$ are not constants in relation (1.13). We will consider the simplest case when $\mathbf{c} = \mathbf{a}$. Then it follows from (1.13) that $\varphi^\cdot = \psi^\cdot$. The integrals (1.10) take the form

$$2\varphi^\cdot(A\mathbf{a}\cdot\boldsymbol{\nu} + A\boldsymbol{\nu}\cdot\boldsymbol{\nu}) = B\boldsymbol{\nu}\cdot\boldsymbol{\nu} - 2(\boldsymbol{\lambda}\cdot\boldsymbol{\nu}) + 2k$$
$$\varphi^{\cdot 2}[A\boldsymbol{\nu}\cdot\boldsymbol{\nu} + A\mathbf{a}\cdot\mathbf{a} + 2(A\mathbf{a}\cdot\boldsymbol{\nu})] = 2(E + \mathbf{s}\cdot\boldsymbol{\nu}) - C\boldsymbol{\nu}\cdot\boldsymbol{\nu} \tag{5.1}$$

Eliminating $\varphi^\cdot$ from relations (5.1) we obtain the special case $A\mathbf{a}\cdot\boldsymbol{\nu} + A\boldsymbol{\nu}\cdot\boldsymbol{\nu} = 0$, which occurs under the following conditions

$$A_{11} = A_{22}, \quad A_{12} = A_{13} = A_{23} = 0, \quad a_0 = A_{11}/(A_{11} - A_{33}) \tag{5.2}$$

Then the right-hand side of the first equation in (5.1) is equal to zero for all φ. This leads to the equalities

$$B_{12}=0,\quad B_{11}=B_{22},\quad \lambda_1=B_{13}a_0,\quad \lambda_2=B_{23}a_0$$
$$2k=2\lambda_3a_0-{}^1\!/_2\, a_0'^2\,(B_{11}+B_{22})-B_{33}a_0^2 \tag{5.3}$$

Putting $\varphi^{\cdot}=\psi^{\cdot}$ in Eq. (1.11) and using (5.1) for $\varphi^{\cdot 2}$, we require the resultling equation to be an identity in φ. Using (5.3) this gives the following conditions

$$\begin{aligned} &B_{12}=B_{13}=B_{23}=0,\quad B_{11}=B_{22},\quad \lambda_1=\lambda_2=0\\ &C_{23}=C_{12}=0,\quad C_{11}=C_{22},\quad s_2=0\\ &s_1=C_{13}\,(4A_{11}A_{33}-A_{11}^2-A_{33}^2)(A_{11}-A_{33})^{-1}\,(2A_{33}-A_{11})^{-1}\\ &\lambda_3=[A_{11}B_{33}+B_{11}\,(A_{33}-2A_{11})]\,(A_{11}-A_{33})^{-1}\end{aligned} \tag{5.4}$$

Hence when conditions (5.2) and (5.4) are satisfied, Eqs (1.1) and (1.2) admit of the solution

$$\begin{aligned} &\omega=\varphi^{\cdot}(\mathbf{a}+\nu),\quad \nu=(a_0'\sin\varphi,\quad a_0'\cos\varphi, a_0),\quad \varphi^{\cdot 2}=a+b\sin\varphi\\ &a=[s_3\,(A_{11}-A_{33})+A_{11}\,(C_{11}-C_{33})]\,(A_{11}-A_{33})^{-2}\\ &b=2\,C_{13}\,[A_{33}\,(A_{33}-2A_{11})]^{1/2}\,(2A_{33}-A_{11})^{-1}(A_{11}-A_{33})^{-1}\end{aligned} \tag{5.5}$$

Solution (5.5) describes a new class of precessional-isoconic motions, where the gyrostat precession is a precession of general form.

We will consider the reduction of the problem to quadratures. In the second equation of (5.5) we introduce the new variable $\beta=\varphi-\pi/2$. Then $\beta^{\cdot}=a+b\cos\beta$ and

$$t=\int_{\beta_0}^{\beta}\frac{d\beta}{\sqrt{a+b\cos\beta}} \tag{5.6}$$

Consequently, we find $\beta(t)$ by inverting an elliptic integral which can be reduced to an elliptic integral of the first kind

$$u=F(\gamma,k)=\int_0^{\gamma}\frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

(k is the modulus), where the method of reduction depends on the values of a and b. We find the angular velocity components from (5.5)

$$\omega_1=\beta^{\cdot}a_0'\cos\beta,\quad \omega_2=-\beta^{\cdot}a_0'\sin\beta,\quad \omega_3=\beta^{\cdot}(1+a_0) \tag{5.7}$$

Case a = b > 0. From (5.6) we obtain

$$\sin\frac{\beta}{2}=\frac{e^{\sqrt{2at}}-1}{e^{\sqrt{2at}}+1}$$

Since $\beta=\pi$ is a stationary point, let us take $\beta=0$ as the initial value of β at $t=0$. When $t\to\infty$ we have $\beta\to\pi$. Obviously, the motion of the body is asymptotic to rest.

Case a > b > 0. From (5.6) we obtain

$$\begin{aligned}&\beta=2\,\mathrm{am}\,(\rho_2 t),\quad \sin\beta=\mathrm{sn}(2\rho_2 t,k_2)\\ &\cos\beta=\mathrm{cn}\,(2\,\rho_2 t,k_2)\end{aligned} \tag{5.8}$$

$$\beta^{\cdot} = 2\rho_2 \operatorname{dn}(\rho_2 t, k_2), \quad \rho_2 = {}^1\!/_2\sqrt{a+b}, \quad k_2 = \sqrt{\frac{2b}{a+b}}$$

The components of the vector $\boldsymbol{\omega}$ are given by relations (5.7), while the components of the vector $\boldsymbol{\nu}$ are

$$\nu_1 = a_0' \cos\beta, \quad \nu_2 = -a_0' \sin\beta, \quad \nu_3 = a_0 \tag{5.9}$$

Since under conditions (5.8) solution (5.7) is periodic with period $2T$, where

$$T_2 = \frac{K}{\rho_2}, \quad K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}}$$

while the stationary hodograph is congruent to the moving one [9], the motion of the body is periodic with period $2T_2$.

Case $b \geqslant |a| > 0, -\arccos(-a/b) \leqslant \beta \leqslant \arccos(a/b)$. If we introduce an auxiliary variable

$$\beta^* = \arcsin\sqrt{\frac{\beta(1-\cos\beta)}{a+b}}$$

then from (5.6) we obtain $\beta^* = \operatorname{am}(p_3 t)$, where $p_3 = \sqrt{b/2}$. Here

$$\sin\beta = 2k_3 \operatorname{sn}(\rho_3 t, k_3)\operatorname{dn}(\rho_3 t, k_3), \quad k_3 = \sqrt{(a+b)/2b}$$
$$\cos\beta = 1 - [(a+b)/b]\operatorname{sn}^2(\rho_3 t, k_3), \quad \beta^{\cdot} = \sqrt{a+b}\operatorname{cn}(\rho_3 t, k_3) \tag{5.10}$$

When relations (5.10) hold solution (5.7) is periodic with period $4T_3$, where $T_3 = K/\rho_3$, K is a complete elliptic integral of the first kind. These properties are also possessed by the components of the angular velocity vector in the stationary space and the gyrostat motion is periodic with the same period.

Case $a > -b > 0$, $0 \leqslant \beta \leqslant 2\pi$. Here it is convenient to introduce an auxiliary variable δ

$$\sin\delta = \sqrt{\frac{(a-b)(1-\cos\beta)}{2(a+b\cos\beta)}}$$

On the basis of (5.6) we obtain $\delta = am(\rho_4 t)$, where $\rho_4 = \sqrt{(a-b)/2}$. Consequently

$$\sin\beta = \sqrt{\frac{a+b}{a-b}}\,\frac{\operatorname{sn}(2\rho_4 t, k_4)}{\operatorname{dn}^2(\rho_4 t, k_4)}$$

$$\cos\beta = \frac{a-b-2a\operatorname{sn}^2(\rho_4 t, k_4)}{(a-b)\operatorname{dn}^2(\rho_4 t, k_4)}$$

$$k_4 = \sqrt{\frac{-2b}{a+b}}, \quad \beta^{\cdot} = \frac{\sqrt{a+b}}{\operatorname{dn}(\rho_4 t, k_4)}$$

On the basis of (5.7) and (5.9) these relations enable us to find $\boldsymbol{\omega}$ and $\boldsymbol{\nu}$ and to conclude that the motion of the gyrostat is periodic with period $T_4 = K/\rho_4$.

To construct the moving hodograph (and hence the stationary one as well) it is sufficient to represent it as the line of intersection of the cone

$$\omega_1^2 + \omega_2^2 - \frac{a_0'^2}{(1+a_0)^2}\omega_3^2 = 0$$

with the cylindrical surface with generators parallel to the $O\omega_3$ axis and intersecting the $O\omega_1\omega_2$ plane along a curve given in polar coordinates ρ, ξ by

$$\rho = a_0' \sqrt{a + b\cos\xi}, \quad (a_0' > 0)$$

The form of the hodograph obviously depends on the values of the parameters a and b.

6. PRECESSIONAL-ISOCONIC MOTIONS IN THE CLASSICAL PROBLEM

Precessional-isoconic motions in the case when $\lambda = 0$ and the matrices B and C are non-zero are of special interest.

If the precessional-isoconic motions are regular, it follows from relations (2.1) that

$$A_{11} = A_{22}, \ A_{12} = A_{13} = A_{23} = 0, \quad s_1 = s_2 = 0$$
$$n^2 A_{33} - a_0 n^2 (A_{11} - A_{33}) + s_3 = 0$$

i.e. this type of motion is only possible in the special case of the Lagrange solution.

To determine the conditions for semiregular professional-isoconic motions of the first type to exist we turn to the results obtained in [9, 4]. The former shows that for the classical problem semiregular precession of the first type only occurs in the Hess solution. The latter shows that there are no isoconic motions in this solution.

When the gyrostat motions are semiregular precessional-isoconic motions of the second type, relations (4.5) must be satisfied. Substituting $B_{ij} = 0$, $C_{kl} = 0$, into them, we obtain a contradiction.

Suppose that the isoconic motion is ia precession of general form. It has been shown [9] that a necessary condition for precession of the general type about the vertical to exist is the vanishing of the constant of the integral of the moment of momentum. From the first relation of (5.1) it follows that in this case the expression in front of φ vanishes for all φ. This reduces to the case considered in Section 5. However, it follows from (5.5) that the precession is regular.

Thus, in the classical problem of the motion of a rigid body in a gravitational field, only regular precessional-isoconic motions of the Lagrange gyrostat about the vertical, exist.

REFERENCES

1. FABBRI R., Sopra una soluzione particolare delle equazioni del moto di un solido pesante intorno ad un punto fizzo. *Atti Accad. Naz. Lincei Cl. Sci. Fiz. met. e. natar.* **19**, 6, 407–415, 495–502, 872–873, 1934.
2. KHARLAMOV P. V., *Lectures on Rigid Body Dynamics.* Izd. Novosibirsk. Univ., Novosibirsk, 1965.
3. KHARLAMOVA Ye. I. and MOZALEVSKAYA G. V., Investigation of the Steklov solution of the equations of motion of a body with a fixed point. *Mat. Fizika* **5**, 194–202, 1968.
4. VARKHALEV Yu. P. and GORR G. V., Isoconic motions of a rigid body with a fixed point. *Mekh. Tverd. Tela* **14**, 20–33. Nauk. Dumka, Kiev, 1982.
5. KHARLAMOVA Ye. I., On the motion of a body with a fixed point. *Mekh. Tverd. Tela* **2**, 35–37, Nauk. Dumka, Kiev, 1970.
6. VERKHOVOD Ye. G. and GORR G. V., A class of isoconic motions in rigid body dynamics with a fixed point. *Mekh. Tverd. Tela* **22**, 33–38, Nauk. Dumka, Kiev, 1990.
7. ORESHKINA L. N., Mathematical analogies of some problems in rigid body dynamics. *Mekh. Tverd. Tela* **18**, 103–110. Nauk. Dumka, Kiev. 1986.
8. YAKH'YA Kh. M., New solutions of the problem of the motion of a gyrostat in potential and magnetic fields. *Vestn. MGU, Ser. 1, Mat. Mekh.* **5**, 60–63, 1985.

9. GORR G. V., ILYUKHIN A. A., KOVALEV A. M. and SAVCHENKO A. YA., *Non-linear Analysis of the Behaviour of Mechanical Systems.* Nauk. Dumka, Kiev, 1984.
10. GORR G. V. and KURGANSKII N. V., On regular precession about the vertical in a problem of rigid body dynamics. *Mekh. Tverd. Tela* **19,** 16–20, Nauk. Dumka, Kiev, 1987.
11. MOZALEVSKAYA G. V. and ORESHKINA L. N., Non-nutational motion of a rigid body. *Mekh. Tverd. Tela* **23,** 1–5, Nauk. Dumka, Kiev, 1991.

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